Exercise 6 Stochastic Models of Manufacturing Systems 4T400, 2 June

1. A pick-and-place machine is mounting electronic components on printed circuit boards. The time (in hours) till failure of the pick-and-place machine is decribed by a random variable X with density $f(x) = cxe^{-\frac{1}{2}x}$ for $x > 0$.

   (a) Calculate the constant c.

   (b) What is the probability that the time to failure of the pick-and-place machine is more than 4 hours?

   (c) Calculate the expected time to failure.

   Note: The primitives of the functions xe^{ax} and x^2e^{ax} are given by

$$\int xe^{ax}dx = \left(\frac{1}{a}x - \frac{1}{a^2}\right)e^{ax}, \quad \int x^2e^{ax}dx = \left(\frac{1}{a}x^2 - \frac{2}{a^2}x + \frac{2}{a^3}\right)e^{ax}.$$

   Answer:

   (a)

$$\int_0^\infty cxe^{-\frac{1}{2}x}dx = 4c = 1,$$

   so $c = \frac{1}{4}$.

   (b)

$$P(X > 4) = \int_4^\infty \frac{1}{4}xe^{-\frac{1}{2}x}dx = 3e^{-2} = 0.406.$$

   (c)

$$E(X) = \int_0^\infty \frac{1}{4}x^2e^{-\frac{1}{2}x}dx = 4 \text{ hour}.$$

2. Packets arrive at a network interface according to a Poisson stream with a rate of $\frac{1}{8}$ packets per time unit. Two types of packets can be distinguished: short packets, which are acknowledgements and they constitute 40% of the incoming packets, and long data packets. The time to transmit a short packet is exactly 1 time unit, to transmit a long one takes exactly 10 time units. Packets are transmitted in order of arrival.

   (a) What is the mean waiting time of an arbitrary packet?

   (b) Short packets are given priority over long data packets. Transmission of packets may not be interrupted. Determine the mean waiting time of a high-priority short packet and a low-priority long one.

   Answer:

   (a) Denote the short packets by type 1, the long ones by type 2. Then

$$\lambda_1 = \frac{1}{8}\cdot\frac{2}{5} = \frac{1}{20}, \quad B_1 = 1, \quad \rho_1 = \frac{1}{20}, \quad \lambda_2 = \frac{1}{8}\cdot\frac{3}{5} = \frac{3}{40}, \quad B_2 = 10, \quad \rho_2 = \frac{3}{4}.$$

   Hence,

$$E(W) = \frac{\rho_1 \cdot \frac{1}{2} + \rho_2 \cdot 5}{1 - \rho_1 - \rho_2} = \frac{151}{8} = 18\frac{7}{8}.$$

(b)

$$E(W_1) = \frac{\rho_1 \cdot \frac{1}{2} + \rho_2 \cdot 5}{1-\rho_1} = \frac{151}{38} = 3\frac{37}{38}, \quad E(W_2) = \frac{\rho_1 \cdot \frac{1}{2} + \rho_2 \cdot 5}{(1-\rho_1)(1-\rho_1-\rho_2)} = \frac{755}{38} = 19\frac{33}{38}.$$

3. Jobs arrive at a machine according to a Poisson process with a rate of 24 jobs per hour. The processsing time is uniform on $[1,3]$ minutes. Jobs are processed in order of arrival.

   (a) Determine the mean flow time of an arbitrary job.

   (b) Small jobs (with a processing time less than 2 minutes) are processed with priority over big jobs (with a processing time greater than 2 minutes). Jobs in process at the machine can not be interrupted. Determine the mean flow time of a samll job, big job and an arbitrary job.

Answer:

(a) We have $\lambda = 2/5$ (job/min), $E(B) = 2$ (min), $E(B^2) = 13/3$ (min^2), so

$$E(R) = \frac{E(B^2)}{2E(B)} = \frac{13}{12} \text{ (min)}.$$

Then, with $\rho = \lambda E(B) = 4/5$, we get for the mean flow time

$$E(S) = \frac{\rho E(R)}{1-\rho} + E(B) = 19/3 = 6.33 \text{ (min)}.$$

(b) Denote the small jobs by type 1 jobs and th ebig ones by type 2. Then $\lambda_1 = \lambda_2 = \frac{1}{5}$ (job/min), $E(B_1) = 3/2$ (min) and $E(B_2) = 5/2$ (min). Since $\rho_1 = \lambda_1 E(B_1) = 3/10$, we get

$$E(S_1) = \frac{\rho E(R)}{1-\rho_1} + E(B_1) = \frac{115}{42} = 2.74 \text{ (min)}.$$

For type 2 jobs we find

$$E(S_2) = \frac{\rho E(R)}{(1-\rho_1)(1-\rho)} + E(B_2) = \frac{365}{42} = 8.69 \text{ (min)}.$$

Thus, for an arbitrary job,

$$E(S) = \frac{1}{2}E(S_1) + \frac{1}{2}E(S_2) = 5.72 \text{ (min)}.$$

Hence, the overall mean flow time is smaller than the mean flow time in (a)!

4. Jobs arrive at a machine according to a Poisson process with a rate of λ jobs per hour. The machine processes the jobs at an exponential rate of μ jobs per hour ($\mu > \lambda$). Whenever the number of jobs exceeds K, an extra identical machine is immediately turned on and starts processing, and when the number of jobs gets down to K again, the one just finishing a job, is turned off.

   (a) Let p_n denote the probability of n jobs in the system. Show that

$$p_n = \begin{cases} p_0 \rho^n, & n = 0, \ldots, K, \\ p_0 \rho^K \left(\frac{\rho}{2}\right)^{n-K}, & n = K, K+1, \ldots \end{cases}$$

where $\rho = \frac{\lambda}{\mu}$ and

$$\frac{1}{p_0} = 1 + \rho + \cdots + \rho^{K-1} + \frac{\rho^K}{1-\frac{\rho}{2}}.$$

(b) Let L denote the number in the system (so $P(L=n)=p_n$). Show that

$$E(L) = p_0 \left(\rho + 2\rho^2 + \cdots + (K-1)\rho^{K-1} + \rho^K \left[\frac{K}{1-\frac{\rho}{2}} + \frac{\frac{\rho}{2}}{(1-\frac{\rho}{2})^2} \right] \right).$$

(c) Suppose $\lambda = 3$ and $\mu = 3\frac{1}{3}$. Compute the mean flow time $E(S)$ in case the second machine is never used (so $K=\infty$), and compute (the reduction of) the mean flow time $E(S)$ for $K=10$.

(d) Compute the rate (average number of times per hour) at which an extra machine is turned on, for $K=10$.

Answer:

(c) For $K=\infty$ (so the system is an $M/M/1$), we have $E(S)=3$ hours. For $K=10$ we get $E(S)=1.428$ hours, a reduction of more than 50%!.

(d) This rate is $p_K\lambda$. For $K=10$ we get that on average $p_{10}3 = 0.146$ times per hour an extra machine is turned on (so approximately once every 7 hours).

5. Consider a job shop consisting of 3 workstations. Workstation i has a single exponential machine processing at rate μ_i, $i=1,2,3$. Three types of jobs arrive at the job shop. Type 1 jobs have to visit workstation 1, 2 and 3 (in that order), type 2 jobs have to visit workstation 1 and 3, and type 3 jobs have to visit workstation 2 and 3. Type i jobs arrive according to a Poisson stream with rate λ_i, $i=1,2,3$.

(a) What is the total arrival rate at each of the workstations, and what are the conditions under which the job shop is stable?

(b) Assume the job shop is stable. Calculate the probability $p(n_1,n_2,n_3)$ of n_1 jobs in workstation 1, n_2 in workstation 2 and n_3 in workstation 3.

(c) Suppose $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 3$. Further, $\mu_1 = 4$, $\mu_2 = 6$ and $\mu_3 = 7$. Calculate the total number in the system and the mean total flow time for type 1, 2 and 3 jobs.

(d) Suppose an additional (identical) machine can be added to one of the workstations. Where would you add a machine to get the greatest reduction of the total number in the system?

Answer:

(a) The total rates are $\lambda_1+\lambda_2$, $\lambda_1+\lambda_3$ and $\lambda_1+\lambda_2+\lambda_3$ in workstations 1, 2 and 3. The job shop is stable if

$$\lambda_1+\lambda_2 < \mu_1, \quad \lambda_1+\lambda_3 < \mu_2, \quad \lambda_1+\lambda_2+\lambda_3 < \mu_3.$$

(b) Let ρ_i be the utilization of workstation i, then

$$\rho_1 = \frac{\lambda_1+\lambda_2}{\mu_1}, \quad \rho_2 = \frac{\lambda_1+\lambda_3}{\mu_2}, \quad \rho_3 = \frac{\lambda_1+\lambda_2+\lambda_3}{\mu_3},$$

and

$$p(n_1,n_2,n_3) = (1-\rho_1)\rho_1^{n_1}(1-\rho_2)\rho_2^{n_2}(1-\rho_3)\rho_3^{n_3}.$$

(c) We have $\rho_1 = \frac{3}{4}$, $\rho_2 = \frac{5}{6}$, $\rho_3 = \frac{6}{7}$, so the mean total number is

$$E(L) = E(L_1)+E(L_2)+E(L_3) = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} + \frac{\rho_3}{1-\rho_3} = 3+5+6 = 14.$$

The mean flow time $E(S)$ is equal to

$$E(S) = \frac{E(L)}{\lambda_1 + \lambda_2 + \lambda_3} = 2\frac{1}{3}.$$

For the mean flow time at each of the workstations we get, by Liitle's law,

$$E(S_1) = \frac{E(L_1)}{\lambda_1 + \lambda_2} = 1, \quad E(S_2) = \frac{E(L_2)}{\lambda_1 + \lambda_3} = 1, \quad E(S_3) = \frac{E(L_3)}{\lambda_1 + \lambda_2 + \lambda_3} = 1.$$

Hence, the mean flow time for type 1 is $E(S_1) + E(S_2) + E(S_3) = 3$, and accordingly, for type 2 and 3, the mean flow time is 2.

(d) Add a machine to workstation 3, which has the highest load. Then workstation 3 acts as an $M/M/2$ with arrival rate 6 and service rate 7 (and thus $\rho_3 = \frac{6}{14}$), so the probability of waiting in workstation 3 is

$$\Pi_3 = \frac{9}{35},$$

so

$$E(W_3) = \frac{\Pi_3}{1 - \rho_3} \frac{1}{2\mu_3} = \frac{9}{280},$$

and

$$E(S_3) = E(W) + \frac{1}{\mu_3} = \frac{7}{40}, \quad E(L_3) = (\lambda_1 + \lambda_2 + \lambda_3)E(S_3) = 1\,\frac{1}{20}(\text{instead of } 6!).$$